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Q. 1 State and prove Logarithmic Test

Soln: Statement: → The positive term series Σu_n converges or diverges according as $\lim_{n\to\infty} \left\{ n \log \frac{u_n}{u_{n+1}} \right\} >$ or < 1.

Proof: Let us compare the given series Σu_n with the auxiliary series Σv_n,

where, $\Sigma v_n = \Sigma \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \frac{1}{(n+1)^p} + \cdots$ to ∞.

We know that Σv_n is convergent or divergent according as $p >$ or < 1.

Hence, by comparison test, the given series Σu_n will be convergent or divergent according as

$$\frac{u_n}{u_{n+1}} > \text{ or } < \frac{v_n}{v_{n+1}}$$

or, $$\frac{u_n}{u_{n+1}} > \text{ or } < \frac{\frac{1}{n^p}}{\frac{1}{(n+1)^p}}$$

or, $$\frac{u_n}{u_{n+1}} > \text{ or } < \left(1+\frac{1}{n}\right)^p$$

or, $$\log \frac{u_n}{u_{n+1}} > \text{ or } < p \log\left(1+\frac{1}{n}\right)$$

or, $$\log \frac{u_n}{u_{n+1}} > \text{ or } < p \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots \text{ to } \infty\right)$$

or, $$n \log \frac{u_n}{u_{n+1}} > \text{ or } < p \left(1 - \frac{1}{2n} + \frac{1}{3n^2} - \cdots \text{ to } \infty\right)$$

Proceeding to limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \left\{ n \log \frac{u_n}{u_{n+1}} \right\} > \text{ or } < p$$

or, $$\lim_{n\to\infty} \left\{ n \log \frac{u_n}{u_{n+1}} \right\} > \text{ or } < 1,$$

which is the logarithmic test.

Q. Q.2. Examine the convergence for $x > 0$:

$$x + \frac{2^2 x^2}{\lfloor 2} + \frac{3^3 x^3}{\lfloor 3} + \frac{4^4 x^4}{\lfloor 4} + \cdots \text{ to } \infty:$$

Soln: Let the nth term of the given series be denoted by u_n. Then, $u_n = \dfrac{n^n x^n}{\lfloor n}$ and $u_{n+1} = \dfrac{(n+1)^{n+1} x^{n+1}}{\lfloor n+1}$

$$= \frac{(n+1)^n (n+1) x^n . x}{\cancel{(n+1)} \lfloor n}$$

$$\therefore \frac{u_{n+1}}{u_n} = \left(1+\frac{1}{n}\right)^n x$$

$$\text{or, } \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n x = ex$$

Hence, by D'Alembert's ratio test, the given series is convergent or divergent according as $ex < 1$ or $ex > 1$, that is, $x < \frac{1}{e}$ or, $x > \frac{1}{e}$

When, $x = \frac{1}{e}$, then this test fails

Now, we apply logarithmic ratio test.

$$\frac{u_n}{u_{n+1}} = \frac{e}{\left(1+\frac{1}{n}\right)^n} \quad \text{or, } \log\frac{u_n}{u_{n+1}} = \log e - n\log\left(1+\frac{1}{n}\right)$$

$$\text{or, } \cancel{n}\log\frac{u_n}{u_{n+1}} = \cancel{n}\left[1 - n\cdot\frac{1}{\cancel{n}} + n\cdot\frac{1}{2n^2} - n\frac{1}{3n^3} + \cdots \text{to } \infty\right]$$

$$= \frac{1}{2} - \frac{1}{3n} + \cdots \text{to } \infty.$$

$$\text{or, } \lim_{n\to\infty}\left(n\log\frac{u_n}{u_{n+1}}\right) = \frac{1}{2} < 1.$$

Hence, by logarithmic ratio test, the given series is divergent when $x = \frac{1}{e}$.